NEAR-CARRIER OSCILLATOR SPECTRUM DUE TO FLICKER AND WHITE NOISE

Gleb V. Klimovitch

Watkins-Johnson Company, Stanford Research Park
3333 Hillview Ave., Palo Alto, CA, 94304-1223
Gleb.Klimovitch@WJ.com, Gleb@LAAS.Stanford.edu

ABSTRACT

The effect of device noise on oscillator phase noise in a close vicinity of average oscillator frequency is analyzed. We show that for 1/f noise, the line shape of a free-running oscillator changes from power to approximately Gaussian function very close to the oscillator frequency. The Gaussian regime occurs when the relationship between device noise and oscillator spectrum is strongly nonlinear. As this relationship approaches linear dependence at higher frequency deviation, there occurs a smooth crossover from Gaussian to well-known power line shape. The commonly used power extrapolation turns out to significantly underestimate phase noise in a part of the near-carrier frequency range.

We also analyze how the near-carrier spectrum of the phase-locked oscillator is affected by the nonlinearity in the dependence of the oscillator spectrum on phase fluctuations. In particular, near-carrier noise suppression of a voltage-controlled oscillator in the second-and-higher-order phase-locked-loop is less than predicted by the linear model or even absent, unless the phase jitter is small.

The proposed theory reduces to existing quantitative models of near-carrier phase noise in oscillators when the relation between noise in the oscillator phase and the output spectral density is linear.

1. INTRODUCTION

Phase noise in oscillators is a crucial limiting factor in many applications, such as wireless communications and frequency standards. It is especially important at frequencies close to the average oscillator frequency ω_0, where significant spectral power is concentrated.

According to [2, 3, 4], under certain conditions, device noise with a power spectral density proportional to $\omega^{-\nu}$ leads to the power shape of oscillator spectrum given by

$$S_V(\omega) \propto |\omega - \omega_0|^{\nu+2} \ . \tag{1}$$

The power law cannot hold as ω approaches ω_0, since the total output power, which is hardly affected by noise, is finite. The deviation from power dependence occurs at frequencies very close to ω_0, where the oscillator output is a nonlinear function of injected noise. For white Gaussian noise $\nu = 0$, the oscillator line shape is Lorentzian [1], and power extrapolation of Eq. (1) overestimates phase noise near ω_0. For purely $f^{-\nu}$ noise, with $\nu \approx 1$, we demonstrate that the resulting oscillator spectrum is approximately Gaussian in a very narrow range around ω_0 between the power tails.

In section 2, we provide both heuristic explanation and rigorous analytical derivation of the line shape of free-running oscillators in the Gaussian and power regions and estimate the crossover frequency between them. Lineshape of locked oscillators is discussed in section 3 followed by a brief summary in section 4. In the Appendix, we specify the frequency range where the oscillator spectrum is determined by the (frequency) modulation noise.

2. PHASE NOISE OF A FREE-RUNNING OSCILLATOR DUE TO NOISE SOURCE WITH POWER SPECTRUM

As shown in the Appendix, there exists a frequency range close to unperturbed oscillator frequency ω_0, in which amplitude noise is negligible in comparison with phase noise, and the oscillator lineshape is determined by the slow random modulation of the moving average of oscillator instantaneous frequency $(\Omega(t) + \omega_0)$. The size of this frequency range is usually determined by the condition that frequency offset $\Delta\omega \equiv \omega - \omega_0$ is small in comparison with both ω_0 and the slowest rate of relaxation processes in oscillator γ_{rel}, such as amplitude relaxation. In tuned oscillators, γ_{rel} is of the order of ω_0/Q, where Q is the quality factor of the resonator, while in low-Q oscillators $\gamma_{rel} \sim \omega_0$. Therefore this frequency range is often defined by:

$$\Delta\omega \ll \omega_0 \ , \ \omega_0/Q \ . \tag{2}$$

(This and other conditions, which usually follow from (2) for practical oscillators but are interesting theoretically, are derived in the Appendix). For example, $\Omega(t)$ can be induced by a variation in parasitic capacitance in the circuit that affects the resonant frequency of the tank, such as gate-to-source capacitance in a FET-based oscillator. For voltagedependent capacitance, such a variation can be induced by slow fluctuations in the short-term average voltage across the capacitance due to flicker noise in the devices, e.g., 1/f noise in the drain current of the FET. In the frequency range defined by Eq. (2), the oscillator spectral density with respect to the carrier is given by the spectrum of the random process

$$V(t) = e^{j\omega_0 t + j\int_0^t \Omega(t_1)dt_1} \equiv e^{j\omega_0 t + j\xi(t)}$$

, where $\xi(t)$ is the *low-frequency* part of the random frequency shift during time interval t. In other words, spectral components of the random frequency shift at frequencies larger than $\Delta\omega$ do not affect the lineshape in the frequency region of Eq. (2) and can be neglected.

A single stationary or cyclostationary source of flicker noise results in fluctuations in $\Omega(t)$ and $\xi(t)$ with power spectral density

$$S_\Omega(\omega) = \frac{K}{\omega^\nu} \ , \ \ S_\xi(\omega) = K\frac{1-\cos(\omega t)}{\omega^{\nu+2}} \ , \qquad (3)$$

where ν is the flicker noise exponent, and K is the proportionality factor.

The linear approaches to phase noise [1, 2, 3, 4], are based on linearizing Eq. with respect to $\xi(t)$, which gives the oscillator spectrum $S_V(\omega)$ that only depends on S_ξ at the translated frequency

$$S_V(\omega) = S_\xi(\omega - \omega_0) \ . \qquad (4)$$

These approaches do not hold at very small frequency offsets $\Delta\omega$ where $\overline{\xi(t\sim 1/\Delta\omega)^2} \geq 1$. For random frequency modulation with power spectrum we obtain from Eq. (3):

$$\overline{\xi^2(t)} = 2K\frac{\cos(\pi(1-\nu)/2)}{\nu(1+\nu)}\Gamma(1-\nu)|t|^{1+\nu} \equiv |\omega_1 t|^{1+\nu} \ , \qquad (5)$$

where Γ is the gamma-function $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$, and the second equality in (5) defines ω_1. Unless $1-\nu \ll 1$, ω_1 is of the order of the frequency crossover between the linear and nonlinear regimes. As ν approaches 1 from below, ω_1 goes to infinity, which results in infinitely broad noise spectrum. This divergence is related with nonstationarity of flicker noise in $\Omega(t)$ at $\nu \geq 1$ that leads to noncyclostationary noise in the oscillator output, an effect not accounted for by the linear models.

In what follows we assume that the noise source is Gaussian, which results in Gaussian $\Omega(t)$. The general expression relating spectra of random Gaussian frequency modulation and oscillator is given by:

$$S_V(\omega) \equiv \int \overline{e^{j\xi(t)}}e^{-j\Delta\omega t}dt = \int e^{-\overline{\xi^2(t)}/2 - j\Delta\omega t} \ . \qquad (6)$$

where the last equality is true because $\xi(t)$ is Gaussian. For white noise, i.e. for $\nu = 0$, we obtain from Eq. (5) that $\overline{\xi^2(t)} \propto |t|$, and Eq. (6) predicts Lorentzian line shape in agreement with [1].

A precisely Gaussian line shape would occur only for $\overline{\xi^2(t)} \propto t^2$, i.e. in the limit $\nu \to 1-0$. However, Eq. (5) implies that as ν approaches 1, the line shape is approximately Gaussian in an increasing frequency range. To prove this, we introduce $\epsilon = 1-\nu$ and substitute $\exp(-\xi^2(t)/2)$ in Eq. (6) by the following expansion:

$$e^{-\frac{\overline{\xi^2(t)}}{2}} = e^{-\frac{|\omega_1 t|^{2-\epsilon}}{2}} = e^{-\frac{(\omega_1 t)^2}{2}}\left(1 + \frac{\epsilon}{2}(\omega_1 t)^2 ln(|\omega_1 t|)\right) \ ,$$

where for our purposes it suffices to keep only the terms up to the first order in ϵ. Then, the oscillator spectrum is given by:

$$S_V(\omega) = \frac{\sqrt{2\pi}}{\omega_1}e^{-\frac{\Delta\omega^2}{2\omega_1^2}} + \Delta S_V(\omega) \equiv S_{V\ Gauss}(\omega) + \Delta S_V(\omega) \ , \qquad (7)$$

where

$$\Delta S_V(\omega) \sim \epsilon \min\left\{\frac{1}{\omega_1} \ , \ \frac{\omega_1^2|ln(\omega_1/\Delta\omega)|}{|\Delta\omega|^3}\right\} \ . \qquad (8)$$

As ϵ approaches zero, there is a growing spectral power range at frequencies around $\Delta\omega = 0$, in which $S_V(|\Delta\omega|) \gg \Delta S_V(|\Delta\omega|)$, and therefore $S_V(|\Delta\omega|)$ is approximately Gaussian. The smooth crossover to non-Gaussian spectrum occurs when $S_{V\ Gauss} \sim \Delta S_V$, which gives frequency deviation $\Delta\omega_c \geq \omega_1$. At extremely small ϵ, $\Delta\omega_c$ is much larger than ω_1.

Now we prove that for 1/f noise, when $\nu \approx 1$, the power extrapolation (3, 4) of oscillator spectrum significantly underestimates spectral density in a large part of the Gaussian region $|\Delta\omega| \leq \Delta\omega_c$. To this end, we estimate the ratio of spectral densities predicted by Eqs. (4, 7) within the Gaussian region at $\Delta\omega = \omega_1$:

$$\frac{S_{V\ Gauss}(\omega_0+\omega_1)}{S_{V\ Power}(\omega_0+\omega_1)} \sim \frac{1/\omega_1}{K/\omega_1^{\nu+2}} \sim \frac{1}{1-\nu} \ , \qquad (9)$$

where the following estimate for ω_1, which can be derived from Eq. (5), is used:

$$\omega_1^{1+\nu} \equiv 2K\frac{\cos\frac{\pi(1-\nu)}{2}}{\nu(1+\nu)}\Gamma(1-\nu) \sim \frac{K}{1-\nu} \ .$$

In Fig. 1, the oscillator spectrum due to purely $1/f^\nu$ noise is calculated from Eqs. (5, 6) and plotted for noise exponents $\nu = 0.95$. The proportionality factor K in Eq. (3) is chosen to yield $\omega_1 = 1$. Also shown power and Gaussian approximations given by Eqs. (5) and (6) respectively. As predicted, the spectrum is approximately Gaussian at $\Delta\omega \leq 1$, while at $\Delta\omega \gg 1$ it asymptotically approaches the power dependence. The calculations confirm that the closer ν to 1, the larger are the ranges of frequency and power over

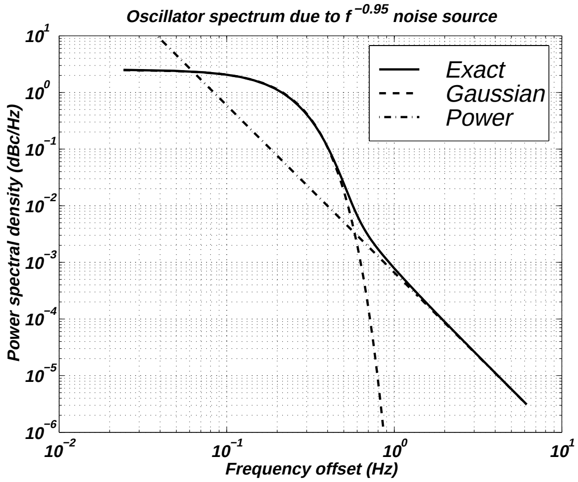

Figure 1: Oscillator spectrum due to $1/f^{0.95}$ noise.

which the line shape is approximately Gaussian. The ratio of $S_{V\ Gauss}$ to $S_{V\ power}$ at frequency offset $\Delta\omega = \omega_1 = 1$ is in agreement with estimate (9).

If the noise source has both flicker and white components, the resulting oscillator spectrum is proportional to the convolution of spectra due to each noise component. The spectral density of fluctuations in $\Omega(t)$ has both flicker and white components: $S_\Omega(\omega) = K/\omega^\nu + S_{\Omega\ white}$, which results in $\overline{\xi^2(t)} = |\omega_1 t|^{1+\nu} + 2S_{\Omega\ white}|t|$. If the white noise is relatively strong in comparison with flicker noise: $S_{\Omega\ white} \geq \omega_1$, then Eq. (6) yields approximately Lorentzian line shape with no flicker region; while linear models incorrectly give both flicker $1/\omega^{\nu+2}$ and white $1/\omega^2$ regions with the phase noise corner at frequency $\Delta\omega \sim (K/S_{\Omega\ white})^{1/\nu}$, where the linear approach is not even applicable.

3. LINE SHAPE OF A LOCKED OSCILLATOR

In this section we consider how nonlinearity in phase-to-voltage conversion affects the output spectrum of a voltage-controlled oscillator (VC0) in a phase-locked loop (PLL). The PLL phase detector (PD) is assumed to be linear. The phase reference is provided by a purely monochromatic source at frequency ω_0.

Let Ω be the pertubation in the instantaneous frequency of the free-running VCO caused by either the control voltage noise or VCO phase noise. For a second-order PLL with the loop filter $F(\omega) = -j/\omega + \tau_z$, the transfer function $H(\omega)$ of Ω to phase error ψ is given by:

$$H \equiv \frac{\psi}{\Omega} = \frac{1}{-j/\omega + K_O K_D F(\omega)} = \frac{j\omega}{\omega_n^2 - \omega^2 + 2j\zeta\omega_n\omega}, \tag{10}$$

where $\omega_n = \sqrt{K_0 K_D}$ is the loop bandwidth, $\zeta = j\omega\tau_z/2$ is the damping ratio, K_O is the VCO gain, and K_D is the PD gain. From Eq. (6) we can obtain the spectrum of the locked VCO as

$$S_V(\omega) = e^{-\overline{\psi^2}}\left[2\pi\delta(\omega-\omega_0) + \int (e^{\overline{\psi(t)\psi(0)}} - 1)e^{j\Delta\omega t}dt\right] \tag{11}$$

where $\overline{\psi^2}$ is the average square of phase jitter, and $\Delta\omega$ stands for the frequency offset $\omega - \omega_0$. The first term in the r.h.s. of Eq. (11) gives the power concentrated at the reference frequency. and will be omitted in what follows. The second term gives the spectrum at $\omega \neq \omega_0$.

In the linear approach to phase-to-voltage conversion, only the first term in the expansion of $(e^{\overline{\psi(t)\psi(0)}} - 1)$ is retained in Eq. (11), which gives the VCO spectrum $\hat{S}_V(\omega) = |H(\Delta\omega)|^2 S_\Omega(\Delta\omega))$ at $\omega \neq \omega_0$. In the linear approximation, the PLL strongly suppresses VCO phase noise at low frequency offsets $\Delta\omega \ll \omega_n$: $\hat{S}_V(\omega \neq \omega_0) \propto \Delta\omega^2$. However, unless the phase jitter is very small: $\overline{\psi^2} \ll 1$, higher-order terms in Eq. (11) need to be taken into account. As a result, there remains less, if any, noise suppression at $\Delta\omega \ll \omega_n$ than predicted by the linear approximation. For moderate jitter $\overline{\psi^2} \leq 1$, Eq. (11) gives at $\omega \neq \omega_0$

$$S_V(\omega) \approx e^{-\overline{\psi^2}}\left[\hat{S}_V(\omega) + \int \frac{d\omega'}{4\pi}\hat{S}_V(\omega')\hat{S}_V(\Delta\omega - \omega')\right] \tag{12}$$

where the last term gives a very important correction to the linear model at low offsets $\Delta\omega \leq \sqrt{(\overline{\psi^2})}/\omega_n$.

In Fig. 2 we plot the spectrum of the VCO in a PLL with $\omega_n = 1$, $\zeta = 1/2$ for five different values of $\overline{\psi^2}$ due to white noise in the VCO. Contrary to the linear approximation, the quadratic rolloff within the loop bandwidth occurs only at sufficiently large frequency offsets $\Delta\omega \geq \sqrt{(\overline{\psi^2})}/\omega_n$, and gradually disappears for jitter $\overline{\psi^2} \geq 1$.

4. SUMMARY

We demonstrate that 1/f noise injected in a free oscillator results in an approximately Gaussian line shape very close to the carrier, as a result of nonlinear conversion of phase fluctuations to oscillator output voltage. The conventional power approximation to the spectrum of the oscillator is proven to considerably underestimate phase noise in a large portion of this frequency region, the corrections being quite significant at power spectral densities ≥ -10dBc.

The same nonlinearity in phase-to-voltage conversion strongly affects the spectrum of locked oscillators at moderate and high phase jitter, even for the linear PLL phase detector. In particular, this nonlinearity significantly deteriorates suppression of VCO noise within the loop bandwidth for jitter exceeding 0.5 squared radians.

The foregoing results are important not only from fundamental viewpoint but also for circuit and system design

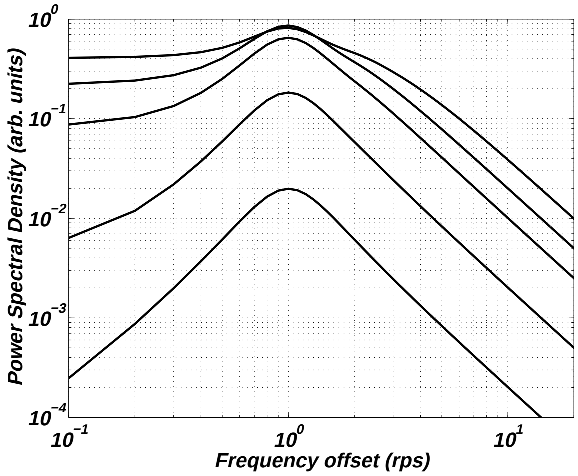

Figure 2: The locked VCO spectrum for phase jitter $\overline{\psi^2}$ of 0.01, 0.1, 0.5, 1, and 2 (bottom to top).

where the near-carrier phase noise is critical.

A. CONVERSION OF INJECTED NOISE TO FLUCTUATIONS OF INSTANTANEOUS FREQUENCY OF OSCILLATIONS

We establish several sufficient conditions on offset frequency $\Delta\omega$ and the spectrum of injected noise, under which the modulation noise dominates over the conversion noise [3], i.e. the effect of the noise source on oscillator spectrum is equivalent to slow modulation of oscillator frequency.

Consider a single noise source and represent it as a cyclostationary Gaussian current noise source decomposed as:

$$i_n(t) = \alpha(\phi(t)) i_{fl}(t) \ , \tag{13}$$

where $i_{fl}(t)$ is a stationary Gaussian noise source with zero mean, which accounts for spectrum of $i_n(t)$ near both DC and harmonics of the oscillator frequency ω_0, and $\alpha(\phi)$ is a periodic function of oscillator phase $\phi(t)$. Constant i_{fl} would induce a constant shift in the average oscillator frequency $\delta\omega$ be proportional to i_{fl} when the latter is sufficiently small: $\delta\omega = \kappa i_{fl}$, where κ is the proportionality factor. If instead $i_{fl}(t)$ changes on the time scale τ_{fl} much larger than the period and the longest relaxation time $\tau_{rel} = 1/\gamma_{rel}$ in oscillator, then $\Omega(t)$, the deviation in the instantaneous oscillator frequency from ω_0 averaged over the period of oscillations, is given by $\Omega(t) = \kappa i_{fl}(t)$. In other words, quasistationary approximation with respect to the slow pertubation $i_{fl}(t)$ applies.

If for frequency offsets larger than given $\Delta\omega$, oscillator output at frequency $\omega_0 + \Delta\omega$ linearly depends on $\Omega(t)$, then the output spectrum $S_V(\omega_0 + \Delta\omega)$ depends on the spectrum of $\Omega(t)$ only at frequencies less than or equal to $\Delta\omega$. As a result, $S_V(\omega_0 + \Delta\omega)$ determined by the random frequency modulation as long as:

$$\Delta\omega \ll \gamma_{rel} \ . \tag{14}$$

A second condition is that phase noise dominates over amplitude noise in oscillator output voltage. Suppose that low-frequency noise source $i_{fl}(t)$ also modulates the amplitude of oscillations, and η is a proportionality factor between $i_{fl}(t)$ and the relative change in amplitude. The amplitude noise can be neglected when

$$\omega - \omega_0 \ll \kappa/\eta \ . \tag{15}$$

For practical oscillators, the ratio of relative changes in amplitude and instantaneous frequency due to low-frequency noise is often of the order of the quality factor Q, then $\kappa/\eta \sim \omega_0/Q$, and conditions (14, 15) are equivalent.

A third condition is that the noise current at frequencies larger than γ_{rel}, for which quasistationary approximation does not apply, should be sufficiently small not to affect oscillator spectrum at smaller frequency offsets $\Delta\omega \ll \gamma_{rel}$. This is the case if the phase shift $\xi_{\omega>\gamma_{rel}}(t)$ due to current noise at frequencies $\omega \geq \gamma_{rel}$ is small at all times, a condition which can be expressed as follows:

$$\overline{\xi^2_{\omega>\gamma_{rel}}(t)} \ \leq \ 4\int_{\gamma_{rel}}^{\infty} S_\Omega(\omega) d\omega/\omega^2 \ll 1 \ . \tag{16}$$

B. REFERENCES

[1] Rytov S.M., Kravtsov Yu.A., and Tatarskii V.I. *Principles of Statistical Radiophysics*, Springer-Verlag, New York, NY, 1978.

[2] Kaertner F.X. ”Analysis of white and $f^{-\alpha}$ noise in oscillators”, *International Journal of Circuit Theory and Applications*, vol.18, 1990, pages 485-519.

[3] Rizzoli, V., Mastri, F., and Masotti, D. ”A general-purpose harmonic-balance approach to the computation of near-carrier noise in free-running microwave oscillators”, *IEEE MTT-S International Microwave Symposium Digest*, pages 309-312, 1993.

[4] Hajimiri A. and Lee T.H. *The Design of Low-Noise Oscillators*, Kluwer Academic Publishers, Boston, MA, 1999.